

## Additive tunnel number and primitive elements

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**Abstract:** It is proven here that if the connected sum of two tunnel number one knots in  $S^3$  is a tunnel number two knot then at least one of the summand knots has a genus two Heegaard splitting with a meridian as a primitive element. Hence this is a necessary and sufficient condition for tunnel number one knots to have additive tunnel number.

### §0. Introduction

The way in which the tunnel number  $t(K)$  of a knot  $K = K_1 \# K_2$  relates to  $t(K_1)$  and  $t(K_2)$  is a long standing question. It was long known that  $t(K_1 \# K_2) \leq t(K_1) + t(K_2) + 1$ . That this inequality is best possible was proved by the author and Rubinstein in [MR] and by Morimoto, Sakuma and Yokota in [MSY]. In the other direction it was shown by Morimoto in [Mo1] that there are prime knots for which  $t(K_1 \# K_2) < t(K_1) + t(K_2)$ . For tunnel number one knots the sum cannot decrease since it was shown by Norwood [No] that tunnel number one knots are prime. It is also known (see Section 1) that if one of the two knots has a genus two Heegaard splitting in which a meridian is primitive then the tunnel number is additive. It remained an open question whether this is also a necessary condition as it is conceivable that the exterior of the connected sum of the knots has a genus two Heegaard splitting which is not induced by any Heegaard splitting of the summand knots.

The main theorem of this paper is the following:

**Theorem 0.1.** *Let  $K_1$  and  $K_2$  be tunnel number one knots in  $S^3$ . Assume that  $t(K) = t(K_1) + t(K_2)$ . Then at least one of  $K_1$  or  $K_2$  has a genus two Heegaard splitting in which a meridian curve represents a primitive element in the handlebody component of the splitting.*

As an immediate corollary we obtain:

**Corollary 0.2.** *Let  $K_1$  and  $K_2$  be tunnel number one knots in  $S^3$ . Then  $t(K) = t(K_1) + t(K_2)$  if and only if one of  $K_1$  or  $K_2$  has a genus two Heegaard splitting in which a meridian curve represents a primitive element in the handlebody component of the splitting.*

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For definitions of the above terminology see Section 1.

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## §1. Preliminaries

In this section we define some of the notions and prove some technical lemmas needed for the proof of the main theorem.

Throughout the paper  $K_1$  and  $K_2$  will be knots in  $S^3$  and  $K = K_1 \# K_2$  will denote the connected sum of  $K_1$  and  $K_2$  (unless specified otherwise). Let  $N()$  denote an open regular neighborhood in  $S^3$ .

Recall that  $(S^3, K)$  is obtained by removing from each space  $(S^3, K_i)$ ,  $i = 1, 2$  a small 3-ball intersecting  $K_i$  in a short unknotted arc and gluing the two remaining 3-balls along the 2-sphere boundary so that the pair of points of  $K_1$  on the 2-sphere are identified with the pair of points of  $K_2$ . If we denote  $S^3 - N(K)$  by  $E(K)$  then  $E(K)$  is obtained from  $E(K_i)$ ,  $i = 1, 2$  by identifying a meridional annulus  $A_1$  on  $\partial E(K_1)$  with a meridional annulus  $A_2$  on  $\partial E(K_2)$ . A knot  $K \subset S^3$  is *prime* if it is not a connected sum of non-trivial knots. The annulus  $A_1 = A_2$  will be denoted by  $A$  and called the *decomposing annulus*.

A *tunnel system* for an arbitrary knot  $K \subset S^3$  is a collection of properly embedded locally unknotted arcs  $t_1, \dots, t_n$  in  $S^3 - N(K)$  so that  $S^3 - N(K \cup t_1 \cup \dots \cup t_n)$  is a handlebody.

Given a tunnel system for a knot  $K \subset S^3$  note that the closure of  $N(K \cup t_1 \cup \dots \cup t_n)$  is always a handlebody denoted by  $V_1$  and the handlebody  $S^3 - N(K \cup t_1 \cup \dots \cup t_n)$  will be denoted by  $V_2$ . For a given knot  $K \subset S^3$  the smallest cardinality of any tunnel system is called the *tunnel number* of  $K$  and is denoted by  $t(K)$ .

A compression body  $V$  is a 3-manifold with a preferred boundary component  $\partial_+ V$  and is obtained from a collar of  $\partial_+ V$  by attaching 2-handles and 3-handles, so that the connected components of  $\partial_- V = \partial V - \partial_+ V$  are all distinct from  $S^2$ . The extreme cases, where  $V$  is a handlebody i.e.,  $\partial_- V = \emptyset$ , or where  $V = \partial_+ V \times I$ , are admitted. Alternatively we can think of  $V$  as obtained from  $(\partial_- V) \times I$  by attaching 1-handles to  $(\partial_- V) \times \{1\}$ . An annulus in a compression body will be called a *vertical annulus* if it has its boundary components on different boundary components of the compression body.

Given a knot  $K \subset S^3$  a *Heegaard splitting* for  $E(K)$  is a decomposition of  $E(K)$  into a compression body  $V_1$  containing  $\partial E(K)$  and a handlebody  $S^3 - \text{int}(V_1)$ . Hence, a tunnel system  $t_1, \dots, t_n$  in  $S^3 - N(K)$  for  $K$  determines a Heegaard splitting of genus  $n + 1$  for  $E(K)$ .

Given a Heegaard splitting  $(V_1, V_2)$  for  $S^3 - N(K_1 \# K_2)$  we can assume that the decomposing annulus  $A$  intersects the compression body  $V_1$  in two vertical annuli  $A_1^*, A_2^*$  and a collection of disks  $D_1, \dots, D_l$ . Note also that  $A$  intersects  $V_2$  in a connected planar surface.

Let  $\mathcal{E} = \{E_1, \dots, E_{t(K)+1}\}$  be a complete meridian disk system for  $V_2$ , chosen to minimize the intersection  $\mathcal{E} \cap A$ . Since  $V_2$  is a handlebody it is irreducible and we can assume that no component of  $\mathcal{E} \cap A$  is a simple closed curve. Furthermore each arc of intersection  $\alpha$  in  $\mathcal{E} \cap A$  is an essential arc.

When we cut  $E(K)$  along the decomposing annulus  $A$  any Heegaard splitting  $(V_1, V_2)$  induces Heegaard splittings on both of  $E(K_1)$  and  $E(K_2)$ : Set  $V_1^i = (V_1 \cap E(K_i)) \cup N(A)$  and  $V_2^i = V_2 - N(A)$ . The pair  $(V_1^i, V_2^i)$  is a Heegaard splitting for  $E(K_i)$ .

We say that an element  $x$  in a free group  $F_n$  is *primitive* if it belongs to some basis for  $F_n$ . A curve on a handlebody  $H$  is *primitive* if it represents a primitive element in the free group  $\pi_1(H)$ . An annulus  $A$  on  $H$  is *primitive* if its core curve is primitive. Note that a curve on a handlebody is primitive if and only if there is an essential disk in the handlebody intersecting the curve in a single point.

Two Heegaard splittings  $(V_1^i, V_2^i)$  for  $E(K_i)$  repectfully, induce a decomposition of  $E(K)$  into  $(V_1, V_2)$ . We obtain  $V_1$  by gluing the compression bodies  $V_1^1$  and  $V_1^2$  along two vertical annuli and  $V_2$  by gluing  $V_2^1$  and  $V_2^2$  along a meridional annulus. Hence  $V_1$  is always a compression body but  $V_2$  is a handlebody if and only if the meridional annulus is a primitive annulus.

Following some ideas of Morimoto (see [Mo 2]) we consider now the planar surface  $P = A \cap V_2$ . It has two distinguished boundary components coming from the vertical annuli  $A_1^*, A_2^*$  and denoted by  $C_1^*, C_2^*$  respectively. There are exactly  $d$  other boundary components of  $P$  which we denote by  $C_1, \dots, C_d$ . With this notation we have  $\partial D_i = C_i$ . The arcs of  $E \cap A$  are contained in  $P$  and come in three types:

- 1 . An arc  $\alpha$  of Type I is an arc connecting two different boundary components of  $P$ .
- 2 . An arc  $\alpha$  of Type II is an arc connecting a single boundary component of  $P$  to itself so that the arc does not separate the boundary components  $C_1^*, C_2^*$ .

3 . An arc  $\alpha$  of Type III is an arc connecting a single boundary component of  $P$  to itself. with the additional property that the arc does separate the boundary components  $C_1^*, C_2^*$ .

Since the  $A$  was chosen to minimize the number of disks in  $A \cap V_1$  then  $P$  is incompressible in the handlebody  $V_2$ . Hence there is a sequence of boundary compressions of  $P$  along disjoint arcs  $\alpha_i$  using sub-disks of  $\mathcal{E}$  so that the end result is a collection of disks. Any such sequence defines an order on the arcs  $\alpha_i$  .

**Definition 1.1.** We call  $\alpha_i$ , an arc of intersection of  $P \cap E$ , a  $d$  – arc if  $\alpha_i$  is of type I and there is some component  $C$  of  $\partial P - (C_1^* \cup C_2^*)$  which meets  $\alpha_i$  and does not meet any  $\alpha_j$  for any  $j < i$ . If  $\alpha_i$  is of type I and connects  $C_1^*$  to  $C_2^*$  it is called an  $e$  – arc

Any outermost arc  $\alpha_i$  determines a sub-disk  $\Delta$  on some  $E_i$  where  $\partial\Delta = \alpha_i \cup \beta$  and  $\beta$  is an arc on  $\partial V_1 = \partial V_2$ . When we perform an isotopy of type A i.e., pushing  $P$  through  $\Delta$  as in [Ja], we produce a band  $b$  with core  $\beta$  on  $\partial V_1 = \partial V_2$ . The following crucial result is proved in [Mo2] pp 41- 42, and [Oc]:

**Theorem 1.2.** (Morimoto) If the decomposing annulus is chosen to minimize the number of components of  $V_1 \cap A$  and  $V_1 \cap A \neq \emptyset$  then in  $V_2 \cap E = P \cap E$ :

- (a) there are no  $d$ -arcs.
- (b) there are no  $e$ -arcs.
- (c) there are no arcs of type II.
- (d) each component  $C \subset \partial P$  has an arc  $\alpha$  of type III with end points on  $C$ .

Consider now a Heegaard splitting  $(V_1, V_2)$  for  $E(K)$  the exterior of  $K = K_1 \# K_2$ , where  $\partial E(K) \subset V_1$  and the decomposing annulus  $A$  meets  $V_1$  in disks and two vertical annuli. Since the annulus  $A$  meets  $V_2$  in a connected planar surface  $P$  it separates  $V_2$  into two components each of which is a handlebody. We will denote the handlebodies  $cl(V_2 - A) \cap E(K_i)$  by  $V_2^i$  respectively. However  $V_1 - A$  might have many components.

**Definition 1.3.** A component of  $cl(V_1 - A)$  which is disjoint from  $\partial E(K_i)$  and intersects  $A$  in  $n$  disks will be called a  $n$ -float.

**Remark:** Note that a n-float is either a 3-ball or a handlebody if its spine is not a tree. Furthermore there is always exactly two components of  $cl(V_1 - A)$  not disjoint from  $\partial E(K_i)$

(one in each of  $E(K_1)$  and  $E(K_2)$ ) and each one is a handlebody of genus at least one as  $V_1$  is a compression body with a  $T^2$  boundary. We denote these special components by  $N_1$  and  $N_2$  depending on whether they are contained in  $E(K_1)$  or  $E(K_2)$  respectively.

Consider now  $E_i \subset \mathcal{E}$  any one of the meridian disks of  $V_2$ . On  $E_i$  we have a collection of arcs corresponding to the intersection with the decomposing annulus. These arcs, as indicated in Fig. 1 below, separate  $E_i$  into sub-disks where disks on opposite sides of arcs are contained in opposite sides of  $A$  i.e., in  $E(K_1)$  or  $E(K_2)$  respectively. So each sub-disk is contained in either  $E(K_1)$  or  $E(K_2)$ . The boundary of these sub-disks is a collection of alternating arcs  $\cup(\alpha_i \cup \beta_i)$  where  $\alpha_i$  are arcs on  $A$  and  $\beta_i$  are arcs on some component of  $cl(V_1 - A)$ .

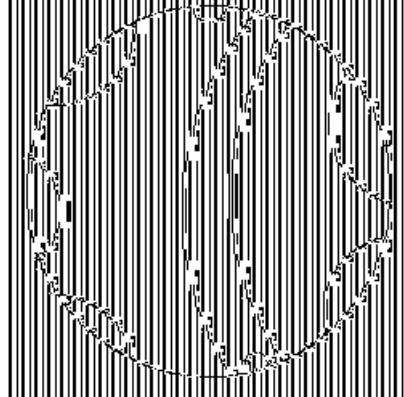


Fig. 1

**Proposition 1.4.** *Let  $K_1$  and  $K_2$  be knots in  $S^3$  and let  $K, A, \mathcal{E}$  be the connected sum, minimal intersection decomposing annulus and meridional system for some Heegaard splitting of  $E(K)$  as above. Then*

- (a) *the  $\beta$  arc part of the boundary of an outermost sub-disk in  $E$  cannot be contained in a n-float which has no genus.*
- (b) *if the  $\beta$  arc part of the boundary of an outermost sub-disk in  $E$  is contained in a  $N_i$  component  $i = 1$  or  $2$  the genus of  $N_i$  is greater than one.*

**Proof.** Denote an outermost sub-disk of some  $E_j$  by  $\Delta$  and suppose it is cut off by an arc  $\alpha$  on  $A$  with end points of a disk  $D_i$  which belongs to some n-float which has no genus.

Further assume  $\partial\Delta = \alpha \cup \beta$  where  $\beta$  is an arc on the n-float meeting  $D_i$  in exactly two points  $\partial\beta = \partial\alpha$ . On  $\partial D_i$  there is a small arc  $\gamma$  so that  $\gamma \cup \beta$  is a simple closed curve on the n-float bounding a disk  $D$  there, since the n-float has no genus (see Fig. 2 below). Furthermore  $\gamma \cup \alpha$  is a simple closed loop on  $A$  which bounds a sub-annulus of  $A$ . Hence  $\gamma \cup \alpha$  bounds a disk  $D'$  on the decomposing 2-sphere of  $K$  intersecting  $K$  in a single point. Thus we obtain a 2-sphere  $D \cup \Delta \cup D'$  which intersects the knot  $K$  in a single point. This is a contradiction which finishes case (a).

For case (b), assume that the outermost disk  $\Delta$  is contained in  $N_1$ , say, and that genus  $N_1$  is one. As before we have  $\partial\Delta = \alpha \cup \beta$  where  $\beta$  is an arc on  $N_1$  and a small arc  $\gamma$  so that  $\gamma \cup \beta$  is a simple closed curve on  $N_1$ . If  $\gamma \cup \beta$  bounds a disk in  $N_1$  we have the same proof as in case (a). If  $\gamma \cup \beta$  does not bound a disk on  $N_1$  we consider small sub-arcs  $\beta_1$  and  $\beta_2$  of  $\beta$  which are respective closed neighborhoods of  $\partial\beta$ . These arcs together with a small arc  $\delta$  on  $\partial N_1 - \partial E(K_1)$  and  $\gamma$  bound a small band  $b$  on  $\partial N_1$ . Notice that  $b \cup_{\beta_1, \beta_2} \Delta$  is an annulus  $A'$ . The annulus  $A'$  together with the sub-annulus  $A''$  of  $A$  cut off by  $\alpha \cup \gamma$  defines an annulus  $A' \cup_{\alpha \cup \gamma} A''$  which determines an isotopy of a meridian curve  $C_1$  to a simple closed curve  $\lambda$  on  $\partial N_1$ . Note that  $N_1$  is a solid torus and  $\pi_1(N_1) = \mathbb{Z}$  which is generated by a meridian  $\mu$  of  $E(K_1)$ . Hence  $[\lambda] = [C_1] = \mu \in \pi_1(N_1)$ . (see Fig. 3). Now we can consider the annulus  $(A - A'') \cup A'$ . If it is non-boundary parallel it is a decomposing annulus with at least one less disk component intersection than  $A$  in contradiction to the choice of  $A$ . If it is boundary parallel we have  $A'' \cup A'$  as a decomposing annulus with a smaller number of disks. Again in contradiction to the choice of  $A$ . So genus  $N_1$  cannot be one and this finishes case (b).

□

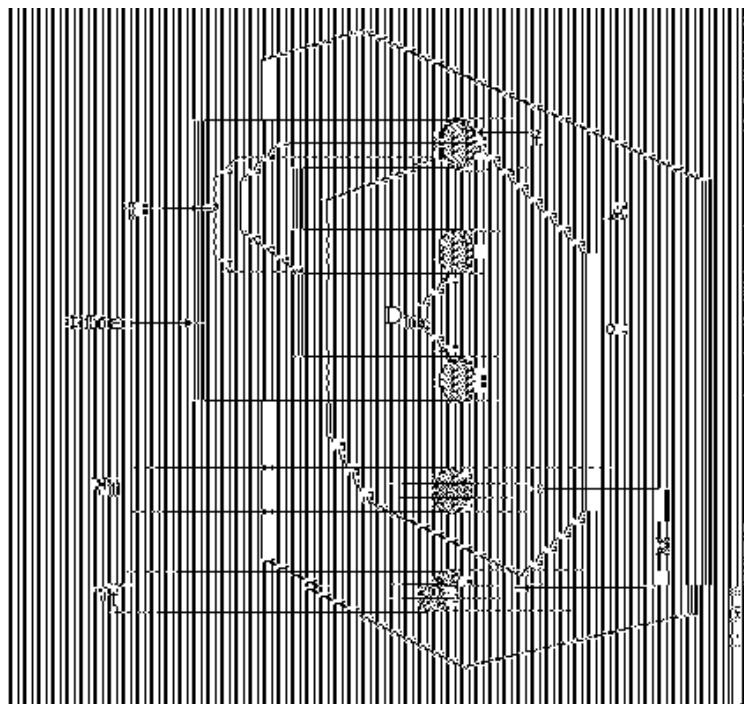


Fig. 2

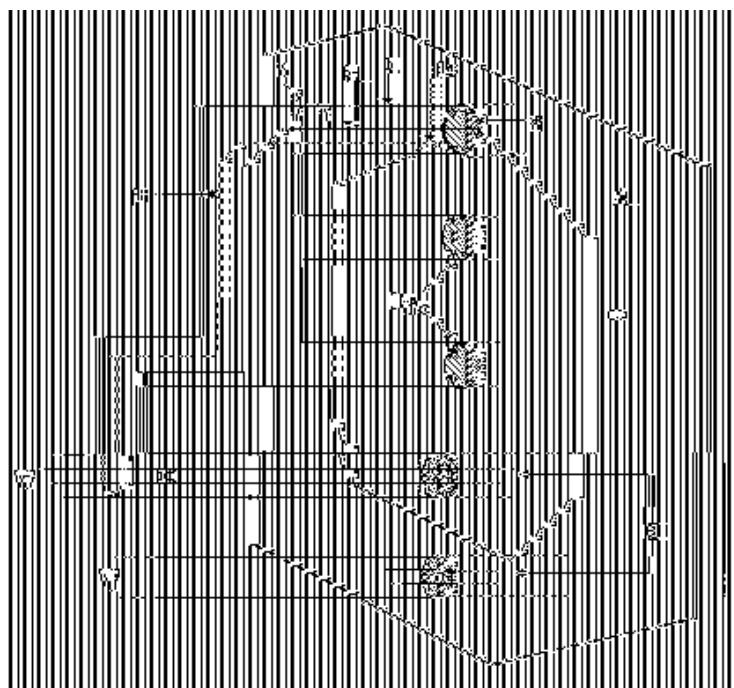


Fig. 3

**Corollary 1.5.** *Every unknotting tunnel system  $\tau$  for  $K = K_1 \# K_2$  must contain at least one tunnel which is disjoint from a decomposing annulus for  $K$  minimizing the number of intersections with  $N(K \cup \tau)$ .*  $\square$

We finish this section with a proposition which is probably due to Morimoto and Sakuma [MS] as it is implicit in their discussion of dual tunnels there (see also [Sc]).

**Proposition 1.6.** *Let  $(V_1, V_2)$  be a Heegaard splitting of a 3-manifold  $M$ . Assume that there is an incompressible annulus  $A \subset V_2$  so that both boundary components of  $A$  are in  $\partial V_2 = \partial V_1$  and one of the boundary components intersects an essential disk  $D$  of  $V_1$ . Then if we add a one handle to  $V_1$  which is a regular neighborhood of an essential arc in  $A$  and remove a regular neighborhood of  $D$  from  $V_1$  we obtain new compression bodies  $V'_1$  and  $V'_2 = M - \text{int}(V'_1)$  defining a new Heegaard splitting of the same genus of  $M$ .*

**Proof.** It is clear that  $V'_1$  is a compression body. To see that  $V'_2$  is a compression body we go through an intermediate step. Cutting  $V_2$  along  $A$  we obtain a compression body as  $A$  is incompressible and since  $\partial A$  meets  $D$  is a single point  $A$  is non-separating. If we add a regular neighborhood of an essential arc  $\nu$  of  $A$  to  $V_1$  we obtain a compression body  $V_1^\#$  of genus one bigger than that of  $V_1$ . Its complement is obtained by gluing together the two copies of the disk  $A - N(\nu)$  on the compression body. Thus getting a compression body  $V_2^\#$  of genus one bigger than that of  $V_2$ . However in the Heegaard splitting  $(V_1^\#, V_2^\#)$  the essential disk  $A - N(\nu)$  intersects the essential disk  $D$  in a single point. So we can reduce the genus by one by removing a regular neighborhood of  $D$  from  $V_1^\#$  and adding it to  $V_2^\#$ . Thus obtaining the compression bodies  $V'_1$  and  $V'_2$  (as indicated in Fig. 4).

$\square$

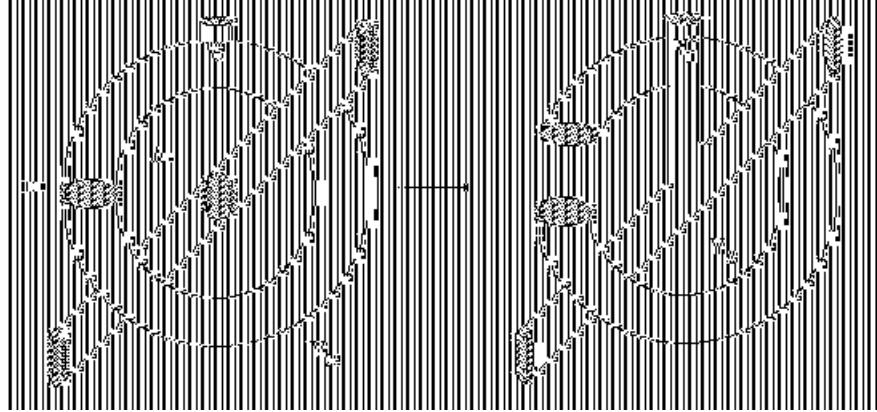


Fig. 4

## §2. Ruling out U-turns

We have the following proposition due to Anna Klebanov (see [Kl]). For the completeness of the argument we present it here with proof.

**Proposition 2.1.** (Klebanov) *Given a decomposing annulus  $A$  which minimizes the number of disk components of  $V_1 \cap A$  then no component of  $cl(V_1 - A)$  is a 3-ball meeting  $A$  in exactly two disks.*

**Remark:** In other words, with the above assumptions no tunnel of  $K$  has U-turns. I.e. no tunnel pierces  $A$  in one direction and then turns around and pierces it again in the opposite direction without meeting any other part of  $V_1$ .

**Proof.** Assume in contradiction that some 2-float is a 3-ball meeting  $A$  in exactly two disks. Denote these disks by  $D_1$  and  $D_2$  but note that these indices do not necessarily agree with the natural order defined on the disks  $D_i$  by Theorem 1.2. We first need the following lemma:

**Lemma 2.2.** *There is some  $E_j \subset \mathcal{E}$  and a sub-disk  $\Delta \subset E_j$  so that  $\partial\Delta = \cup(\alpha_r \cup \beta_s)$ , where the  $\beta_s$  arcs are contained in the 2-float and the  $\alpha_r$  are arcs on  $A$  of type I or III. Hence  $\partial\Delta \subset 2\text{-float} \cup A$ .*

**Proof of Lemma.** By Theorem 1.2 there is some arc  $\alpha$  with end points on  $D_1$ . This arc  $\alpha$  occurs in some  $E_j$  and separates it into two disks. Consider the disk adjacent to  $\alpha$  on

the same side of  $A$  as the 2-float. We can assume that the sub-disk is to the right of  $\alpha$ . This sub-disk cannot be outer-most as the 2-float has no genus. Hence there are more arcs of intersection further to the right of  $\alpha$ . If all arcs on the sub-disk  $\Delta$  adjacent to  $\alpha$ , which are not on  $\partial E_j$ , are of type III the disk  $\Delta$  satisfies the conclusion of the lemma and we are done: Since all arcs of type III have end points on  $D_1$  or  $D_2$  the  $\beta$  arcs must be contained in the 2-float.

So we assume that some arc to the right of  $\alpha$  is of type I. If all such arcs have both end points on  $D_1$  or  $D_2$  we are done as well. The only way in which  $\partial\Delta$  can leave the 2-float is by means of an arc of type I. Hence if the lemma fails there are at least two arcs of type I with exactly one end point not on  $D_1$  or  $D_2$ . Since  $\partial\Delta$  is connected and if we leave the 2-float by means of an arc of type I we must come back to it also by means of an arc of type I. Consider such an arc  $\rho$ , it cannot be outer-most as it is of type I and would then be a d-arc contradicting Theorem 1.2. Hence further to the right there is some arc  $\alpha'$  of type III with end points on  $D_1$  or  $D_2$ . One of the two disks adjacent to  $\alpha'$  is on the 2-float. Assume that it is  $\Delta'$  and that it is on the right of  $\alpha'$ . Then we start the argument again with  $\alpha'$ . This procedure must end since the intersection is finite.

Assume therefore that  $\Delta'$  is to the left of  $\alpha'$ . If all arcs in  $\partial\Delta' - \partial E_j$  are either all of type I or type III with end points on  $D_1$  or  $D_2$  we are done as before. If there an arc of type I with no end points on  $D_1$  or  $D_2$  then there is an arc  $\rho'$  of type I with exactly one end point on  $D_1$  or  $D_2$ . It cannot be outermost as before so farther out there is an arc of type III with end points on  $D_1$  or  $D_2$ . So we can start the argument with  $\rho'$ . However the procedure must terminate as the intersection is finite. Hence at some stage we obtain a disk  $\Delta$  with  $\partial\Delta - \partial E_j$  consisting of arcs of type I or type III all of which have end points on  $D_1$  or  $D_2$ . Hence all the  $\beta$  arcs are on the 2-float. (see Fig.1)

□

Consider now an essential sub-annulus  $A'$  of  $A$  containing the disks  $D_1$  and  $D_2$ . It is a meridional annulus in  $(S^3, K)$  so we can cap off  $A'$  by two meridian disks  $D_1^*$  and  $D_2^*$  in  $(S^3, K)$  to obtain a 2-sphere intersecting  $K$  in exactly two points in  $D_1^*$  and  $D_2^*$ . If we attach the boundary of the 2-float to this 2-sphere along  $D_1$  and  $D_2$  we get a 2-torus  $T$ . By the above lemma  $\partial\Delta$  is contained in  $T$ .

**Lemma 2.3.** *The loop  $\partial\Delta$  is essential in  $T$ .*

**Proof of Lemma.** Assume that  $\partial\Delta$  bounds a disk  $\Delta'$  on  $T$ . If  $\Delta'$  contains only one of  $D_1^*$  or  $D_2^*$  then the 2-sphere  $\Delta' \cup \Delta$  intersects  $K$  in a single point. If  $\Delta'$  contains both of  $D_1^*$  and  $D_2^*$  then the 2-sphere  $\Delta' \cup \Delta$  is a decomposing 2-sphere for  $K$  as if it was boundary parallel we could isotope  $A$  off  $\Delta$  reducing the intersection of  $A$  and  $\mathcal{E}$  in contradiction. The 2-sphere  $\Delta' \cup \Delta$  meets  $V_1$  in less disks than  $A$  as  $\Delta$  is in  $V_2$  and  $\Delta'$  does not contain  $D_1$  or  $D_2$  in contradiction to the choice of  $A$ .

Since  $\partial\Delta$  bounds a disk  $\Delta'$  on  $T$  the intersection of  $\partial\Delta$  with a core curve of the meridional annulus  $A'$  is even. Similarly the intersection of  $\partial\Delta$  with the boundary of a cocore disk of the 2-float is even. Hence the number of arcs of type I is even and so is the number of  $\beta$  arcs (these are the arcs which intersect the boundary of a cocore disk of the 2-float). Hence the number of arcs of type III (the  $\alpha$  arcs) is also even. As a consequence the disk  $\Delta'$  is a union of bands glued together to each other at their ends. The bands correspond to the areas in  $A$  between the arcs of type I and between the arcs of type III and also on the 2-float between the  $\beta$  arcs (see Fig. 5).

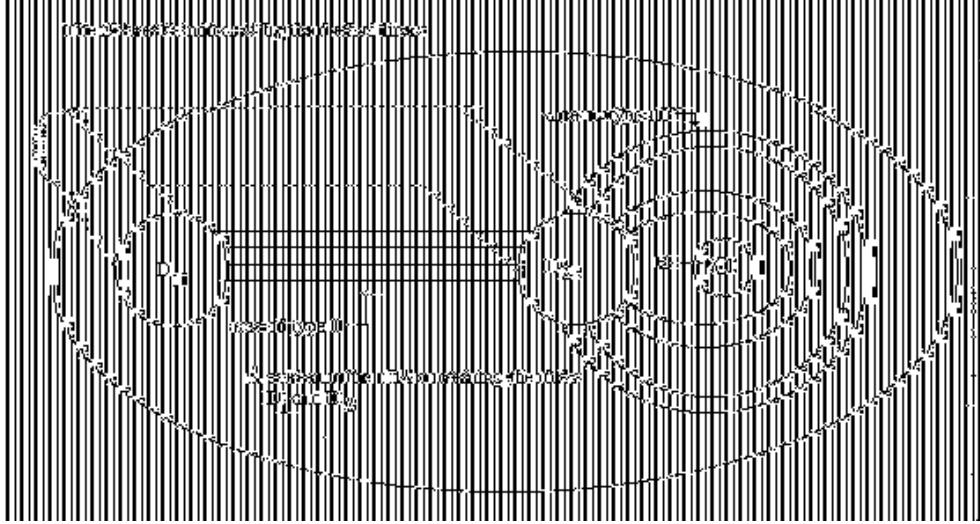


Fig. 5

Since the bands are glued to each other along small arcs on both ends, the number of gluing arcs is equal to the number of bands. So an Euler characteristic argument shows that

$$\chi(\Delta') = \sum \chi(\text{bands}) - \sum \chi(\text{gluing arcs}) = 0$$

But this is obviously a contradiction and hence  $\partial\Delta$  is essential in  $T$ .

□

Now we do 2-surgery on  $T$  along the curve  $\partial\Delta$  by removing an annulus neighborhood of  $\partial\Delta$  on  $T$  and gluing two copies of  $\Delta$ . By an Euler characteristic argument we obtain a 2-sphere intersecting  $K$  in two points on  $D_1^*$  and  $D_2^*$ . If we remove  $D_1^*$  and  $D_2^*$  we obtain an annulus  $A''$ . We can now replace the annulus  $A'$  by the annulus  $A''$  and get a new decomposing annulus  $(A - A') \cup A''$  which does not intersect the disks  $D_1$  and  $D_2$  (since  $A''$  does not). This contradicts the choice of  $A$  and hence we cannot have 2-floats of genus zero as stated in the proposition.  $\square$

### §3. Additive tunnel number one knots

In this section we assume that both  $K_1$  and  $K_2$  are tunnel number one knots and hence are both prime by a result of Norwood (see [No]). We further assume that  $t(K) = t(K_1) + t(K_2) = 2$ . We have the following lemma:

**Lemma 3.1.** *Let  $K_1$  and  $K_2$  be tunnel number one knots in  $S^3$  so that  $t(K) = t(K_1) + t(K_2) = 2$ . Let  $(V_1, V_2)$  be a genus three Heegaard splitting of  $S^3 - N(K)$ . If  $A$  is a decomposing annulus minimizing intersection with  $V_1$  then we can choose a spine for  $V_1$  which is  $\partial E(K) \cup t_1 \cup t_2$  where  $t_1 \cap A = \emptyset$  and  $N(t_2) \cap A$  is either empty or is composed of at most two disks. As indicated in Fig. 6 (a), (b), (c) and (d).*

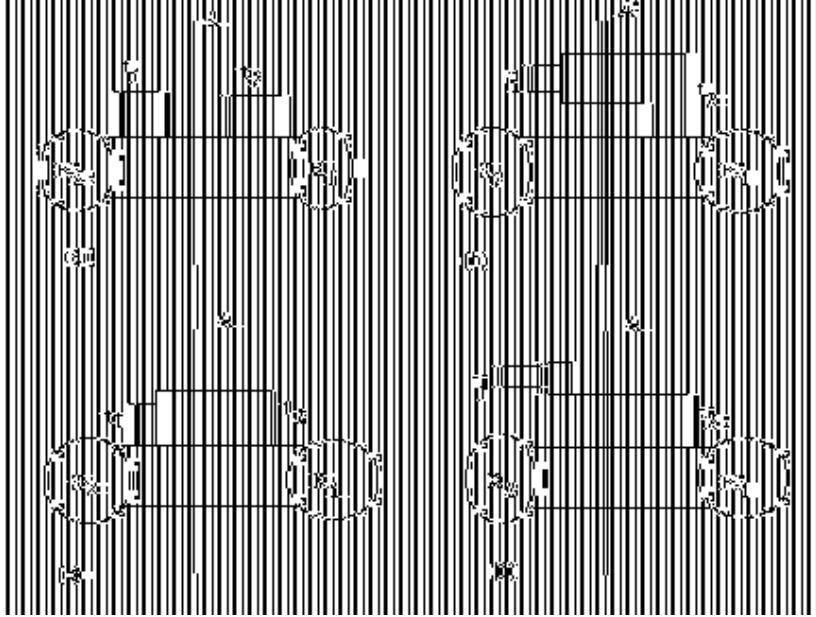


Fig. 6

**Proof.** By Proposition 1.4 there is a tunnel  $t_1$  i.e., a maximal cocore arc of an essential disk in  $V_1$  which does not meet  $A$  at all. If we cut  $V_1$  along the essential disk we have a genus two compression body. We can now choose a spine for the 1-handle connected to the  $T^2 \times I$  part of the compression body. Denote this arc by  $t_2$ . It cannot intersect  $A$  in more than two points as this would create 2-floats of genus zero in contradiction to Proposition 2.1. Hence there are four possibilities.

1. The arc  $t_2$  has both its end points on one side of  $A$  and does not meet  $A$ . In this case  $t_1$  is on the other side of  $A$ . Since both knots are tunnel number one knots.
2. The arc  $t_2$  has both its end points on one side of  $A$  and does meet  $A$ . In this case  $t_2$  meets  $A$  in two points and  $t_1$  has both of its end points on  $t_2$ . As otherwise  $t_2$  would create a 2-float of genus zero or if  $t_1$  had one end point on  $t_2$  and the other on  $K$  then  $V_1 - A$  would have exactly two components both of genus one in contradiction to Proposition 1.4.
3. The arc  $t_2$  has one end point on one side of  $A$  and the other on the other side. In this case  $t_2$  meets  $A$  in a single point as otherwise we have genus zero 2-floats. The arc  $t_1$  can either have both its end points on  $t_2$  or one end point on  $t_2$  and the other on  $K$ .
4. The arc  $t_2$  has one end point on one side of  $A$  and the other on the other side on  $t_2$  creating a little loop. In this case  $t_1$  must have both its end points on  $t_2$  or otherwise we are in case 3.

All these cases are indicated in Fig. 6

□

We are now ready to prove the theorem.

**Proof.** By Lemma 3.1 we only need to consider the four possible configurations of Cases 1 – 4 discussed there.

Case 1: Since  $A$  is incompressible it cuts  $V_2$  into two handlebodies  $V_2^1$  and  $V_2^2$ . These handlebodies when glued along  $A$  yield a handlebody. Hence  $A$  must be primitive in one of them.

Case 2: When we cut  $V_2$  along  $A$  we obtain a 2-float of genus one intersecting the annulus  $A$  in two disks denoted by  $D_1$  and  $D_2$ . We can assume that the 2-float, which we denote by  $V$  is contained in  $E(K_2)$ . Any outermost disk  $\Delta$  in any meridian disk of  $V_2$  must have  $\partial\Delta = \alpha \cup \beta$  where  $\alpha$  is an arc of type III and  $\beta$  is an arc on  $V$ . The shared end points of

$\alpha$  and  $\beta$  are either on  $D_1$  or  $D_2$  and we can assume that they are on  $D_1$  as the picture is symmetric (see Fig. 7, below).

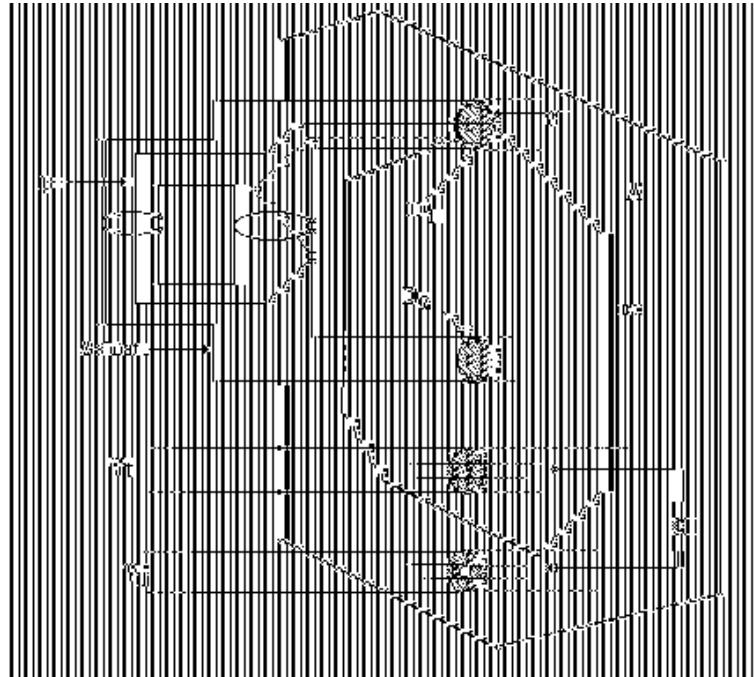


Fig. 7

Hence we have a solid torus  $V$  with two marked disks  $D_1$  and  $D_2$  on it and there is an essential arc  $\beta$  connecting  $D_1$  to itself (see Fig. 8).

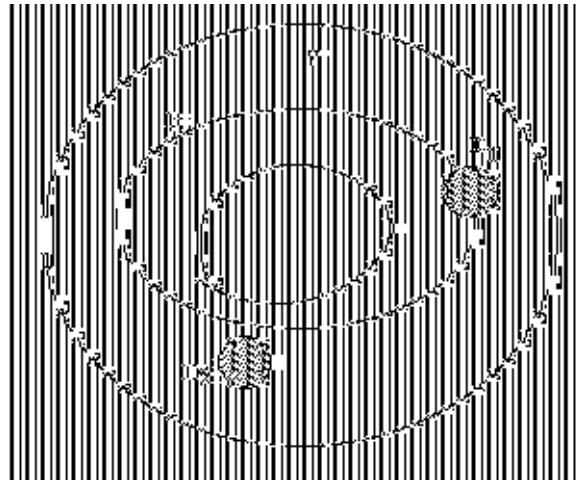


Fig. 8

Claim: There is a meridian disk  $D$  of  $V$  which intersects  $\beta \cup D_1$  in a single point.

*Proof of Claim:* Consider a small arc  $\gamma$  on  $\partial D_1$  so that  $\gamma \cup \alpha$  is a meridian curve of  $E(K_2)$ , chosen so that  $\gamma \cup \alpha$  does not separate the disk  $D_1$  from  $D_2$ . As before, consider small sub-arcs  $\beta_1$  and  $\beta_2$  of  $\beta$  which are respective closed neighborhoods of  $\partial\beta$ . These arcs together with a small arc  $\delta$  on  $\partial V - (\partial D_1 \cup \partial D_2)$  and  $\gamma$  bound a small band  $b$  on  $\partial V$ . Notice that  $b \cup_{\beta_1, \beta_2} \Delta$  is an annulus  $A'$ . The annulus  $A'$  together with the sub-annulus  $A''$  of  $A$  cut off by  $\alpha \cup \gamma$  define an annulus  $A' \cup_{\alpha \cup \gamma} A''$  which determines an isotopy of a meridian curve  $C_1 \subset \partial A$  to a simple closed curve  $\lambda = \beta - (\beta_1 \cup \beta_2) \cup \delta$  on  $\partial V$ . So  $\lambda$  is homotopic in  $V$  to some power of a core curve  $v$  of  $V$  (i.e.,  $\lambda = v^n, n \in \mathbb{Z}$ ). Hence in  $E(K_2)$  a meridian curve  $\mu = \alpha \cup \gamma \simeq v^n$ . Since  $[\mu]$  is a generator of  $H_1(E(K_2)) = \mathbb{Z}$  this can only happen if  $n = \pm 1$ . Hence  $\lambda$  is homotopic to a curve on  $\partial V$  which intersects a meridian disk of  $V$  in a single point. As  $\partial V = T^2$ , it follows that  $\lambda$  is isotopic to a longitude curve of  $V$  and hence there is a disk isotopic to a standard meridian disk of  $V$  which intersects  $\lambda$  in a single point. This finishes the proof of the Claim.

□

The annulus  $A_1 = A' \cup A''$  has the following properties:

- (i) Both boundary components are on  $\partial V_1 = \partial V_2$ .
- (ii) The interior of  $A_1$  is in  $V_2$ .
- (iii) There is an essential disk in  $V_1$  intersecting  $A_1$  in a single point.

Hence  $A_1$  satisfies all the conditions for Proposition 1.6 and we can change the Heegaard splitting  $(V_1, V_2)$  by replacing the tunnel  $t_2$  by a tunnel  $t_3$  which is an essential arc of  $A_1$ . Notice that we can do this so that the new tunnel is slightly pushed off  $A_1$  so it is disjoint from it (see Fig. 4). We now have a new Heegaard splitting  $(V'_1, V'_2)$  for  $E(K)$  and the decomposing annulus  $A$  intersects  $V'_1$  in exactly two disks as the only change took place in  $E(K_2)$  away from  $A$ .

The annulus  $A$  cannot minimize the intersection with  $V'_1$  as  $V'_1 - A$  has exactly two components none of which has genus bigger than one. This contradicts Proposition 1.4. If  $|V'_1 \cap A| = 1$  then we are in Case 3 treated below. If  $|V'_1 \cap A| = 0$  then as in Case 1 the annulus  $A$  is primitive in one of the Heegaard splittings induced on  $E(K_1)$  or  $E(K_2)$  by cutting  $(V'_1, V'_2)$  along  $A$  (as in Section 1).

Case 3: In this case  $A \cap V_2$  is a once punctured annulus. Since  $V_1 \cap E(K_1)$  has only one component of genus one all outermost disks of  $\mathcal{E}$  must be contained in  $E(K_2)$ . As before, consider an outermost disk  $\Delta$  in some meridian disk  $E$  of  $V_2$ . We have  $\partial\Delta = \alpha \cup \beta$  where  $\alpha$  is an arc of type III on  $A$  and  $\beta$  is an arc on  $\partial V_1 - A$ .

Notice that in our situation  $(V_1^1, V_2^1)$ , the induced Heegaard splitting, is a genus two Heegaard splitting of  $E(K_1)$  and  $(V_1^2, V_2^2)$  is a genus three Heegaard splitting of  $E(K_2)$ .

Now do a boundary compression of  $A$  along the disk  $\Delta$  in  $E(K_2)$  (An isotopy of type A in the terminology of [Ja]). Denote by  $\tilde{A}$  the annulus obtained from  $A$  after this isotopy. This isotopy does not change the Heegaard splitting  $(V_1, V_2)$  but does change the induced Heegaard splittings on  $E(K_i), i = 1, 2$ . The boundary compression removes a regular neighborhood  $N(\Delta)$  from  $V_2^2$  and adds it to  $V_2^1$  along the arc  $\alpha$ . We obtain a new handlebody  $V_2'^1$  of genus two in  $E(K_1)$ . Since we have not changed the genus of the handlebody  $V_2^1$  and cutting along  $\tilde{A}$  does induce a Heegaard splitting on  $E(K_1)$  we obtain a genus two Heegaard splitting of  $E(K_1)$  by taking  $V_1'^1 = E(K_1) - \text{int}V_2'^1$ .

The situation on  $E(K_2)$  is slightly more complicated. The disk  $\Delta \subset V_2^2$  is an essential disk. If it was not essential then  $\Delta$  together with a disk on  $\partial V_2$  bound a 3-ball  $B_0$ . The boundary of  $B_0$  is  $\Delta$ , a disk in  $A$  and a disk on  $\partial V_2$ . The 3-ball  $B_0$  allows us to isotope  $A$  off of  $\alpha$  thus reducing the number of intersections of  $\mathcal{E} \cap A$  in contradiction to the choice of  $\mathcal{E}$  (see Fig. 9).

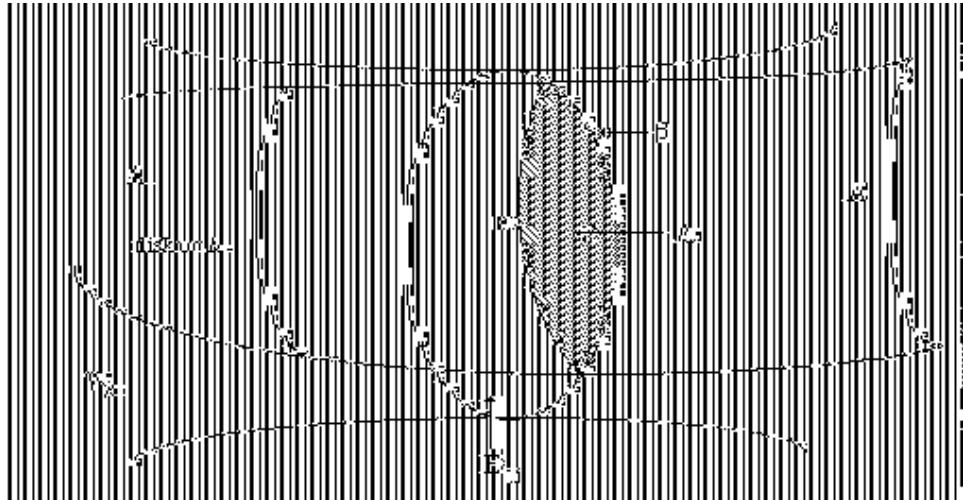


Fig. 9

Since as before we change  $A$  only by an isotopy and cutting along  $\tilde{A}$  induces a Heegaard splitting on  $E(K_2)$  the disk  $\Delta$  is non-separating and after cutting along  $\tilde{A}$  we obtain a handlebody  $V_2'^2$  of genus two in  $E(K_2)$ . Hence the Heegaard splitting  $(V_1'^2, V_2'^2)$  induced on  $E(K_2)$  by cutting along  $\tilde{A}$  is of genus two.

The annulus  $\tilde{A}$  intersects the Heegaard splitting  $(V_1, V_2)$  as follows:  $\tilde{A} \cap V_1$  is two vertical annuli  $A_1^*$  and  $A_2^*$  and one essential annulus,  $\tilde{A} \cap V_2$  is two essential annuli denoted by  $A_1$  and  $A_2$ . Both of  $A_1$  and  $A_2$  are meridional annuli (see Fig. 10). Note that the isotopy of  $A$  can be done in a small neighborhood of the arc  $\alpha$  on  $A$  disjoint from the core curves of  $A_1$  and  $A_2$ . Hence on each of the handlebodies  $V_2'^1, V_2'^2$ , in the induced Heegaard splittings of  $E(K_1)$  and  $E(K_2)$ , we have copies of the meridional annuli  $A_1$  and  $A_2$ .

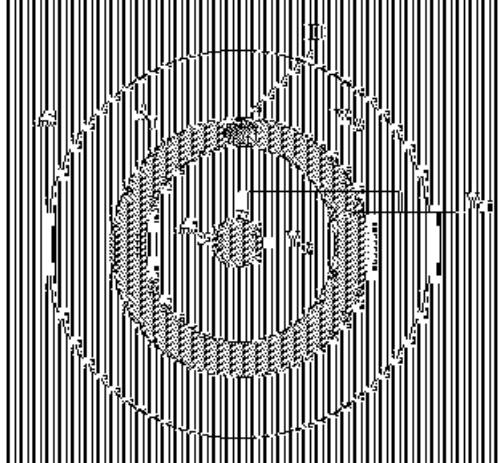


Fig. 10

Assume now that no genus two Heegaard splitting of  $E(K_1)$  has a primitive meridian curve. Since gluing  $V_2'^1$  to  $V_2'^2$  along the annuli  $A_1$  and  $A_2$  yields a handlebody  $V_2$  both annuli  $A_1$  and  $A_2$  must be primitive on  $V_2'^2$ . However a priori they need not be primitive simultaneously.

Lemma 2.3.2 of [CGLS] states that two primitive curves on a genus two handlebody are either simultaneously primitive or when we add two disks to the handlebody along the curves (i.e., 2-surgery along the curves) we obtain a non-trivial punctured Lens space. In our situation the two curves in question are the core curves of the meridional annuli  $A_1$  and  $A_2$ . Hence if the curves are not simultaneously primitive then 2-surgery along them yields a Lens space. But  $K_2$  is a knot in  $S^3$  and 2-surgery along meridian curves yields  $S^3$  in contradiction.

Recall that  $V_2'^2$  was obtained from  $V_2^2$  by cutting  $V_2^2$  open along the disk  $\Delta$ . So in order to obtain  $V_2^2$  back we need to identify the two copies of  $\Delta$  on  $\partial V_2'^2$ . Since  $\beta \subset \partial\Delta$  and  $\beta$  is parallel to an arc, also denoted by  $\beta$ , in  $\partial A_i$  this identification will result in identifying the primitive annuli  $A_1$  and  $A_2$  on  $\partial V_2'^2$  along the arc  $\beta$  (see Fig. 11). Two annuli identified along an arc yield a once punctured annulus. In our situation this planar surface has two boundary components corresponding to simultaneously primitive curves on  $V_2'^2$  and the third, which is the result of the identification, corresponds to  $\partial D$  where  $D$  is the single disk of  $V_1 \cap A$ .

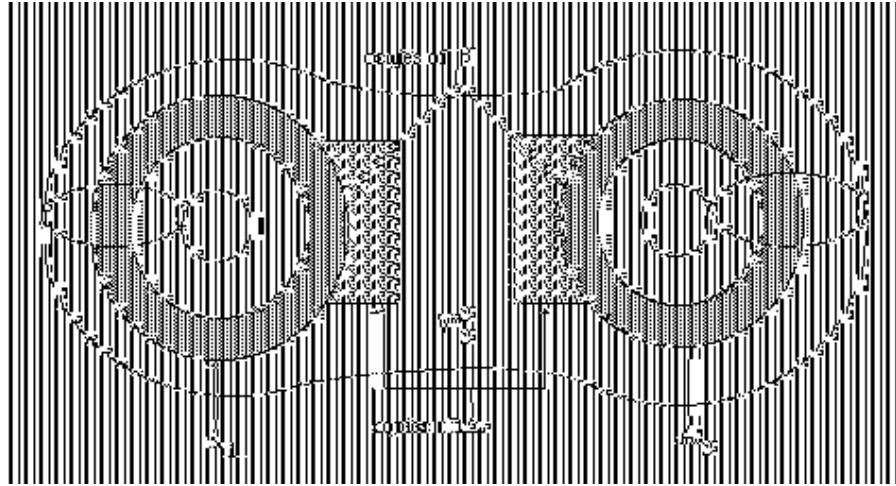


Fig. 11

Denote the generators of  $\pi_1(V_2'^2)$  corresponding to the core curves of the primitive annuli  $A_1$  and  $A_2$  by  $x$  and  $y$  respectively. Identifying the two copies of  $\Delta \subset \partial V_2'^2$  “creates the third genus” of  $V_2^2$  and induces an HNN extension of  $\pi_1(V_2'^2)$  where the new generator resulting from the identification is denoted by  $z$ .

Hence the third puncture in  $A \cap V_2$  (i.e.,  $\partial D$ ) corresponds to the word  $xzy^{-1}z^{-1} \in F(x, y, z) = \pi_1(V_2^2)$ . As both  $x$  and  $y$  are primitive so is  $xzy^{-1}z^{-1}$  since  $\{x, z, xzy^{-1}z^{-1}\}$  is a basis for  $F(x, y, z)$ .

We conclude that  $\partial D$  is a primitive element in  $\pi_1(V_2^2)$  and so must intersect an essential disk of  $V_2^2$  in a single point. Hence the Heegaard splitting of  $E(K_2)$  induced by cutting along  $A$  is reducible. Thus if we remove a regular neighborhood  $N(D)$  from  $V_1^2$  and add it to  $V_2^2$  we still have a handlebody and a genus two Heegaard splitting of  $E(K_2)$ . In the fundamental group of this handlebody  $xzy^{-1}z^{-1} = 1$  so  $y = z^{-1}xz \in F(x, z)$ . But

now  $x = [A_1]$  is a primitive element in this new handlebody. Hence we found a genus two Heegaard splitting of  $E(K_2)$  in which a meridional annulus is primitive.

Case 4: The proof in this case is the same as for Case 3.

This finishes the proof of the theorem.  $\square$

**Question:** Is the statement of Theorem 0.1 true for knots with tunnel number bigger than one?

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